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Probability Weights and Measures on Finite Effect Algebras

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We study probability weights and measures on finite effect algebras, thus generalizing the existing theory for orthomodular posets and orthoalgebras. Our development proceeds somewhat more generally in that we study weights and measures associated with an antichain in the positive cone of a euclidean vector space with the standard partial ordering.

1. INTRODUCTION

A foundation for the theory of probability measures (states) on orthomodular lattices, orthomodular posets, and orthoalgebras has been established by Bennett (1968, 1970), D'Andrea *et al.* (1991), Greechie (1971), Greechie and Miller (1970), Gudder (1965, 1988), Hamhalter *et al.* (1995), Kläy (1985), Pták and Pulmannová (1991), Rüttimann (1977a, b, 1980), among others. For reasons given in Greechie and Foulis (1995), there is now considerable interest in even more general structures called *effect algebras* (or *D-posets*), and the question naturally arises whether the existing theory of probability measures on orthoalgebras extends to effect algebras. The purpose of this paper is to indicate that the answer is affirmative and to further develop and consolidate the resulting theory. Although our results apply primarily to finite effect algebras, we shall formulate our theory in the somewhat more general context of so-called *T-weights*.

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2. NOTATION AND TERMINOLOGY

The ring of integers and the field of real numbers are denoted by $\mathbb{Z}$ and $\mathbb{R}$, respectively. Also, $\mathbb{R}^+ := \{a \in \mathbb{R} \mid a \geq 0\}$ and $\mathbb{Z}^+ := \mathbb{Z} \cap \mathbb{R}^+$. (The notation $:=$ means equals by definition.) In this paper, all vector spaces are understood to be over the field $\mathbb{R}$ of real numbers.

2.1. Definition. Let V be a vector space and let $M \subseteq V$.

(i) $\mathrm{lin}(M)$ is the linear span of M.
(ii) $\mathrm{aff}(M)$ is the affine span of M.
(iii) $\mathrm{con}(M)$ is the convex hull of M.
(iv) $\mathrm{rank}(M) := \dim(\mathrm{lin}(M))$.
(v) If $M = \mathrm{con}(M)$, then $\dim(M) := \dim(\mathrm{aff}(M))$.

A *positive cone* in a vector space V is a subset V^+ of V such that $V^+ + V^+ \subseteq V^+$, $\mathbb{R}^+V^+ \subseteq V^+$, and $V^+ \cap (-V^+) = \{0\}$. A vector space V with a distinguished positive cone $V^+ \subseteq V$ is called a *partially ordered vector space*. If V is a partially ordered vector space, then the relation $\leq$ defined for λ, $\mu \in V$ by $\lambda \leq \mu$ iff $\mu - \lambda \in V^+$ is a translation-invariant partial order on V. (The abbreviation iff means if and only if.)

2.2. Definition. Let V be a partially ordered vector space.

(i) V^+ *generates* V iff $V = \mathrm{lin}(V^+)$, i.e., iff $V = V^+ - V^+$.
(ii) A linear functional f on V is *positive* iff $0 \leq f(\lambda)$ for all $\lambda \in V^+$.
(iii) $\Omega \subseteq V$ is a *cone base* for V^+ iff $\Omega = \mathrm{con}(\Omega) \subseteq V^+$ and every nonzero element $\mu \in V^+$ can be written uniquely in the form $\mu = a\omega$ with $a \in \mathbb{R}^+$ and $\omega \in \Omega$.
(iv) $\xi \in V^+$ is an *order unit* iff, for every $\lambda \in V$, there exists a positive integer k such that $\lambda \leq k\xi$.
(v) $A \subseteq V$ is an *antichain* iff $\lambda, \mu \in A$ with $\lambda \leq \mu$ implies $\lambda = \mu$.

If X and Y are sets, then Y^X is the set of all functions $f\colon X \to Y$. If $X \neq \emptyset$, then $\mathbb{R}^X$ forms a lattice-ordered vector space (a *Riesz space*) under pointwise operations and with the standard positive cone $(\mathbb{R}^+)^X$. If V is a vector subspace of $\mathbb{R}^X$, then V is partially ordered by the *induced positive cone* $V^+ := V \cap (\mathbb{R}^+)^X$.

2.3. Definition. Suppose that $f \in \mathbb{R}^X$, $M \subseteq \mathbb{R}^X$, and $Y \subseteq X$.

(i) f is *strictly positive* on Y iff $0 < f(y)$ for all $y \in Y$.
(ii) M is *strictly positive* on Y iff, for each $y \in Y$, there is at least one $f \in M$ such that $0 < f(y)$.
(iii) The *support* of f, in symbols $\mathrm{supp}(f)$, is defined by $\mathrm{supp}(f) := \{x \in X \mid f(x) \neq 0\}$.

The notion of a strictly positive set $M \subseteq \mathbb{R}^X$ will have an important role to play in the sequel. Note that f is strictly positive iff the set $\{f\}$ is strictly positive, however a strictly positive subset $M \subseteq \mathbb{R}^X$ need not contain any strictly positive functions at all. If $M \subseteq (\mathbb{R}^+)^X$, then M is strictly positive on Y iff $Y \subseteq \cup_{f \in M} \operatorname{supp}(f)$.

If X is a finite nonempty set, then $\mathbb{R}^X$ is a Euclidean space with the standard inner product $\langle f, g\rangle := \Sigma_{x \in X} f(x)g(x)$ for all $f, g \in \mathbb{R}^X$. In our development, elements of $\mathbb{R}^X$ have dual roles to play. When we are thinking of such elements simply as functions from X to $\mathbb{R}$, we denote them by lowercase Latin letters $f, g, \ldots$. However, when we are thinking of them as corresponding to linear functionals on $\mathbb{R}^X$ determined as usual by the inner product, we denote them by lowercase Greek letters $\nu, \mu, \ldots$. Thus $\nu \in \mathbb{R}^X$ determines the linear functional $f \mapsto \langle \nu, f\rangle$ for all $f \in \mathbb{R}^X$.

3. MOTIVATION

Let L be a finite effect algebra with unit u and let X be a fixed set of generators for L (Bennett and Foulis, 1997; Foulis and Bennett, 1994). By *a relation* for L we mean an equation of the form

$$\oplus_{x \in X}\, t(x)x = u \tag{1}$$

where the coefficients in the orthosum are determined by a function $t : X \to \mathbb{Z}^+$. If $t : X \to \mathbb{Z}^+$ and (1) holds, then t is called a *multiplicity function* for L and the set T of all such functions is called the *total* set of multiplicity functions for L.

Let T be the total set of multiplicity functions for L. Then T is a finite antichain in $(\mathbb{Z}^+)^X \subseteq (\mathbb{R}^+)^X$ and T is strictly positive on X. The elements of the effect algebra L have the form

$$p = \oplus_{x \in X}\, f(x)x \tag{2}$$

where $f \in (\mathbb{Z}^+)^X$ and $f \leq t$ for some $t \in T$. Conversely, if $t \in T$ and $f \in (\mathbb{Z}^+)^X$ with $f \leq t$, then the orthosum in (2) exists and f determines a unique element $p \in L$. All information concerning the structure of the effect algebra L is encoded in the set T (Foulis *et al.*, 1996).

An $\mathbb{R}$-*valued measure* (or *charge*) on L is a mapping $\nu : L \to \mathbb{R}$ such that, for $p, q \in L$, $p \perp q \Rightarrow \nu(p \oplus q) = \nu(p) + \nu(q)$ (Foulis and Bennett, 1994). If ν is an $\mathbb{R}$-valued measure on L and $p \in L$ is given by (2), then

$$\nu(p) = \Sigma_{x \in X}\, \nu(x)f(x) \tag{3}$$

and it follows that ν is determined by its values on X. As a consequence of (1) and (3), we have

$$\nu\,(u) = \Sigma_{x\in X}\, \nu\,(x)\, f\,(x) \tag{4}$$

for all $t \in T$. Conversely, if ν: $X \to \mathbb{R}$ and the expression on the right side of (4) is independent of the choice of $t \in T$, then ν can be extended uniquely in accordance with (3) to an $\mathbb{R}$-valued measure on L. The restriction of an $\mathbb{R}$-valued measure on L to the set of generators X is called a *T-weight*. The T-weights form a vector subspace V of $\mathbb{R}^X$ characterized by the condition that $\nu \in V$ iff $\nu \in \mathbb{R}^X$ and

$$e(\nu) := \Sigma_{x\in X}\, \nu(x)t(x) = \langle \nu, t\rangle \tag{5}$$

is independent of the choice of $t \in T$. The vector space V is understood to be partially ordered by the induced positive cone $V^+ := V \cap (\mathbb{R}^+)^X$. Evidently e is a positive linear functional on V.

An $\mathbb{R}$-valued measure ω on L is called a *probability measure* iff $\omega(p) \in \mathbb{R}^+$ for all $p \in L$ and $\omega(u) = 1$. The T-weights that correspond to the probability measures are precisely those that belong to the convex set $\Omega := V^+ \cap e^{-1}(1)$. We refer to a T-weight $\omega \in \Omega$ as a *T-probability weight*.

In applications of the theory of effect algebras, the elements $x \in X$ represent basic outcomes or effects associated with measurements of observables for a physical system, and each state of the system induces a probability measure on L. It is traditional to assume that every outcome or effect in X has a nonzero probability of manifesting itself in some state—in other words, that Ω is strictly positive on X.

In view of the foregoing, we propose to concentrate our attention on T-weights and T probability weights. In fact it turns out that much of the theory can be developed for the more general situation in which T is contained in the positive cone $(\mathbb{R}^+)^X$ of the real vector space $\mathbb{R}^X$, and that will be our approach in what follows.

4. *T*-WEIGHTS AND *T*-PROBABILITY WEIGHTS

For the the remainder of this paper, we adopt the following notational conventions.

4.1. Standing Notation. X is a finite nonempty set, T is a nonempty subset of $(\mathbb{R}^+)^X$, and $0 \notin T$.

4.2. Definition. An element $\nu \in \mathbb{R}^X$ is called a *T-weight* iff $e_T(\nu) := \langle \nu, t\rangle$ is independent of the choice of $t \in T$. The subset of $\mathbb{R}^X$ consisting of all T-weights is denoted by $V(T)$.

Clearly, $V(T)$ is a vector subspace of $\mathbb{R}^X$ and e_T is a linear functional on $V(T)$. We refer to e_T as the *intensity*, the *variation*, or the *trace functional* *corresponding* to T. If T is understood, we write simply V and e rather than $V(T)$ and e_T.

4.3. Definition. For $x \in X$, the linear functional $f_x : V \to \mathbb{R}$ defined by $f_x(\nu) := \nu(x)$ for all $\nu \in V$ is called the *evaluation functional corresponding* to x.

We regard V as a partially ordered vector space with the induced positive cone $V^+ := V \cap (\mathbb{R}^+)^X$. Note that the evaluation functionals f_x, $x \in X$, and the intensity functional e are positive linear functionals on V. Also $V^+ = \bigcap_{x \in X} f_x^{-1}(\mathbb{R}^+)$ is a polyhedral cone in V, i.e., it is an intersection of finitely many closed half-spaces. Furthermore, the evaluation functionals form a total set of linear functionals on V, i.e., $\bigcap_{x \in X} \ker(f_x) = \{0\}$. Consequently, every linear functional on V is a linear combination of the evaluation functionals. In particular, e can be so expressed.

4.4. Lemma. $t \in T \Rightarrow e = \Sigma_{x \in X}\, t(x) f_x$.

Proof. For $\nu \in V$,

$$(\Sigma_{x \in X}\, t(x) f_x)\,(\nu) = \Sigma_{x \in X}\, t(x)\nu(x) = \langle \nu, t\rangle = e(\nu). \qquad \blacksquare$$

If $T \subseteq (\mathbb{Z}^+)^X$ is the total set of multiplicity functions for an effect algebra L with a finite set of generators X, the equation in Lemma 4.4 is an analogue of Equation (1) in Section 3 with $\oplus$ replaced by Σ, x replaced by f_x, and u replaced by e.

4.5. Definition. $\Omega(T) := (V(T))^+ \cap (e_T)^{-1}(1)$ and an element $\omega \in \Omega(T)$ is called a *T-probability weight*.

Again, if T is understood, we write simply Ω rather than $\Omega(T)$. Obviously Ω is a convex subset of the positive cone V^+ in V. The following examples show that Ω may be empty, it may consist of a single vector, or it may be unbounded.

4.6. Examples. Let $X = \{1, 2\}$ and identify $\psi \in \mathbb{R}^X$ with the vector $(\psi(1), \psi(2)) \in \mathbb{R}^2$. (i) If $T = \{(0, 1), (1, 1), (1, 0)\}$, then $V = \{(0, 0)\}$ and $\Omega = \emptyset$. (ii) If $T = \{(0,2), (1,1)\}$, then $V = \{(a, a) \mid a \in \mathbb{R}\}$ and $\Omega = \{(\frac{1}{2}, \frac{1}{2})\}$. (iii) If $T = \{(1, 0)\}$, then Ω is the unbounded set $\{(1, a) \mid 0 \le a\}$.

4.7. Lemma. If $\Omega \neq \emptyset$, then $B \subseteq T \Rightarrow T \cap \mathrm{lin}(B) \subseteq \mathrm{aff}(B)$.

Proof. Let $\omega \in \Omega$ and let $t \in T \cap \mathrm{lin}(B)$. Then there exist $t_i \in B \subseteq T$ and $a_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n$ such that $t = \Sigma_i a_i t_i$. Therefore $\langle \omega, t\rangle = e(\omega) = 1$ for all $t \in T$, so $1 = \langle \omega, t\rangle = \Sigma_i a_i \langle \omega, t_i\rangle = \Sigma_i a_i$, whence $t \in \mathrm{aff}(B)$. $\blacksquare$

If T is strictly positive on X, the situation in part (iii) of Example 4.6 is ruled out, and in fact Ω is order bounded (hence norm bounded) in $\mathbb{R}^X$.

4.8. Lemma. If T is strictly positive on X, there exists $\phi \in (\mathbb{R}^+)^X$ such that $\omega \in \Omega \Rightarrow 0 \leq \omega \leq \phi$.

Proof. For each $x \in X$, select $t_x \in T$ such that $t_x(x) > 0$ and define $\phi \in \mathbb{R}^X$ by $\phi(x) := 1/t_x(x)$. If $\omega \in \Omega$, we have $1 = e(\omega) = \langle \omega, t_x \rangle \geq \omega(x)t_x(x)$ for all $x \in X$, and it follows that $\omega \leq \phi$. ■

4.9. Lemma. The following conditions are mutually equivalent:

(i) T is strictly positive on X.
(ii) $V^+ \cap \ker(e) = \{0\}$.
(iii) Ω is a cone base for V^+.

Proof. (i) $\Rightarrow$ (ii): Suppose $0 \neq \mu \in V^+ \cap \ker(e)$. Then there exists $x \in X$ with $\mu(x) > 0$. But $0 = e(\mu) = \langle \mu, t \rangle \geq \mu(x)t(x) \geq 0$, so $t(x) = 0$ for all $t \in T$, and T is not strictly positive on X.

(ii) $\Rightarrow$ (iii): Assume (ii) and let $0 \neq \mu \in V^+$. Then $a := e(\mu) > 0$ and $\omega := (1/a)\mu \in V^+ \cap e^{-1}(1) = \Omega$ with $\mu = a\omega$. On the other hand, if $\mu = a\omega$, then $e(\mu) = ae(\omega) = a$ and $\omega = (1/a)\mu$.

(iii) $\Rightarrow$ (i): Assume (iii) and suppose T is not strictly positive on X. Then there exists $X_0 \in X$ such that $t(x_0) = 0$ for all $t \in T$. Let $\mu \in \mathbb{R}^X$ be defined for $x \in X$ by $\mu(x) := 0$ if $x \neq x_0$ and $\mu(x_0) := 1$. Then $0 \neq \mu \in V^+$ with $e(\mu) = 0$. By (iii), there exists $a > 0$ and $\omega \in \Omega$ with $\mu = a\omega$, so $0 = e(\mu) = ae(\omega) = a$, contradicting $a > 0$. ■

If $M \subseteq \mathbb{R}^X$, then $M^\perp := \{\psi \in \mathbb{R}^X | \langle \psi, f \rangle = 0 \text{ for all } f \in M\}$. Also $^{\#}X$ denotes the cardinal number of X.

4.10. Theorem. Let $t_0 \in T$ and let $T - t_0 := \{t - t_0 | t \in T\}$. Then:

(i) $\mathrm{lin}(T - t_0) = V^\perp$.
(ii) $\mathrm{lin}(T - t_0)$ is independent of the choice of $t_0 \in T$.
(iii) $\dim(V^\perp) = \mathrm{rank}(T - t_0) = \dim(\mathrm{aff}(T))$.
(iv) $\ker(e) = T^\perp$.
(v) $\mathrm{rank}(T) + \dim(\ker(e)) = {}^{\#}X$.

Proof. (i) Let $W := \mathrm{lin}(T - t_0)$. Then, for $\nu \in \mathbb{R}^X$, $\nu \in W^\perp$ iff $\langle \omega, t - t_0 \rangle = 0$ for all $t \in T$ iff $\langle \nu, t \rangle = \langle \nu, t_0 \rangle$ for all $t \in T$ iff $\nu \in V$. Therefore, $W^\perp = V$, so $W = W^{\perp\perp} = V^\perp$. Part (ii) follows immediately from (i).

(iii) Because $t_0 \in T$, we have $\mathrm{aff}(T) - t_0 = \mathrm{lin}(T - t_0)$; hence, since the dimension of an affine subspace is invariant under translation,

$$\dim(\mathrm{aff}(T)) = \dim(\mathrm{aff}(T) - t_0) = \dim(\mathrm{lin}(T - t_0))$$
$$= \mathrm{rank}\,(T - t_0)$$

Also, $\dim(V^\perp) = \dim(\mathrm{lin}(T - t_0))$ by (i), and (iii) follows.

(iv) For $\nu \in \mathbb{R}^X$, we have

$$\nu \in T^{\perp} \Leftrightarrow \langle \nu, t \rangle = 0 \text{ for all } t \in T \Leftrightarrow \nu \in V \text{ and } e(\nu) = 0$$
$$\Leftrightarrow \nu \in \ker(e)$$

(v) $\mathbb{R}^X$ is the direct sum of $\mathrm{lin}(T)$ and $T^{\perp} = (\mathrm{lin}(T))^{\perp}$, so

$$\#X = \dim(\mathbb{R}^X) = \dim(\mathrm{lin}(T)) + \dim(T^{\perp})$$
$$= \mathrm{rank}(T) + \dim(\ker(e))$$

by (iv). ■

4.11. Corollary. $e \neq 0 \Rightarrow \dim(V) = \#X + 1 - \mathrm{rank}(T)$.

Proof. By part (v) of Theorem 4.10, $\mathrm{rank}(T) + \dim(\ker(e)) = \#X$ and, if $e \neq 0$, then $\dim(\ker(e)) = \dim(V) - 1$. ■

4.12. Lemma. Suppose $\Omega \neq \emptyset$ and $B \subseteq T \subseteq \mathrm{lin}(B)$. Then $\emptyset \neq B$, $V(T) = V(B)$, $e_T = e_B$, and $\Omega(T) = \Omega(B)$.

Proof. Since $0 \notin T \neq \emptyset$ and $B \subseteq T \subseteq \mathrm{lin}(B)$, it follows that $0 \notin B \neq \emptyset$. Because $B \subseteq T$, we have $V(T) \subseteq V(B)$. Suppose $\phi \in V(B)$ and $t \in T$. Then $t \in T \cap \mathrm{lin}(B) \subseteq \mathrm{aff}(B)$ by Lemma 4.7, so there exist $t_i \in B$ and $a_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n$ such that $t = \Sigma_i a_i t_i$ and $\Sigma_i a_i = 1$. Consequently,

$$\langle \phi, t \rangle = \Sigma_i a_i \langle \phi, t_i \rangle = \Sigma_i a_i e_B(\phi) = e_B(\phi) \Sigma_i a_i = e_B(\phi)$$

whence $\phi \in V(T)$ with $e_B(\phi) = \langle \phi, t \rangle = e_T(\phi)$. Therefore, $V(T) = V(B)$, $e_T = e_B$, and it follows that $\Omega(T) = \Omega(B)$. ■

4.13. Lemma. Suppose $\Omega \neq \phi$ and $T \subseteq S \subseteq \mathrm{aff}(T) \cap (\mathbb{R}^+)^X$. Then, $0 \notin S$, $V(T) = V(S)$, $e_T = e_S$, and $\Omega(T) = \Omega(S)$.

Proof. If $s \in S \subseteq \mathrm{aff}(T) \cap (\mathbb{R}^+)^X$, then there exist $t_i \in T$, $a_i \in \mathbb{R}$ for $i = 1, 2, \ldots n$, with $s = \Sigma_i a_i t_i$ and $\Sigma_i a_i = 1$. Therefore, for $\nu \in V(T)$,

$$\langle \nu, s \rangle = \Sigma_i a_i \langle \nu, t_i \rangle = \Sigma_i a_i e_T(\nu) = e_T(\nu) \tag{1}$$

Thus $0 \notin S$, else we could take $s = 0$ and $\nu \in \Omega$ in (1) to obtain the contradiction $0 = 1$. Furthermore, by (1),

$$\nu \in V(T) \Rightarrow \nu \in V(S) \text{ with } e_s(\nu) = e_T(\nu)$$

Since $T \subseteq S$, we also have $V(S) \subseteq V(T)$, whence $V(T) = V(S)$ and $e_T = e_s$, from which it follows that $\Omega(T) = \Omega(S)$. ■

4.14. Theorem. Suppose $\Omega \neq \emptyset$, let B be a maximal linearly independent subset of T, and let $B \subseteq S \subseteq \mathrm{aff}(T) \cap (\mathbb{R}^+)^X$. Then $0 \notin S$, $V(T) = V(S)$, $e_T = e_s$, and $\Omega(T) = \Omega(S)$.

Proof. Since B is a maximal linearly independent subset of T, we have $B \subseteq T \subseteq \text{lin}(B)$, so $0 \notin B$ and Lemma 4.12 implies that $\emptyset \neq B$, $V(T) = V(B)$, $e_T = e_B$, and $\Omega(T) = \Omega(B)$. Therefore, by Lemma 4.7, $T = T \cap \text{lin}(B) \subseteq \text{aff}(B)$, whence $\text{aff}(T) \subseteq \text{aff}(B)$. Since $B \subseteq T$, we also have $\text{aff}(B) \subseteq \text{aff}(T)$, so $\text{aff}(T) = \text{aff}(B)$. Thus $B \subseteq S \subseteq \text{aff}(B) \cap (\mathbb{R}^+)^X$, so by Lemma 4.13 with T replaced by B, $0 \notin S$, $V(B) = V(S)$, $e_B = e_S$, and $\Omega(B) = \Omega(S)$. Therefore, $V(T) = V(B) = V(S)$, $e_T = e_B = e_S$, and $\Omega(T) = \Omega(B) = \Omega(S)$. ■

If B is a maximal linearly independent subset of T and we are interested only in assessing the structure of the vector space $V(T)$, the intensity functional e_T, and the convex subset $\Omega(T)$, then by Theorem 4.14, we can replace T by any set between B and $\text{aff}(T) \cap (\mathbb{R}^+)^X$. For some purposes, it is useful to enlarge T and for others, to make T smaller.

4.15. Theorem. If $\Omega(T)$ is strictly positive on X, then T is an antichain.

Proof. Suppose $\Omega = \Omega(T)$ is strictly positive on X and that $s, t \in T$ with $s \leq t$ but $s \neq t$. Then $x \in X \Rightarrow 0 \leq t(x) - s(x)$ and there exists $y \in X$ with $0 < t(y) - s(y)$. Since Ω is strictly positive, there exists $\omega \in \Omega$ with $0 < \omega(y)$. However, this leads to the contradiction

$$0 < \omega(y)[t(y) - s(y)] \leq \Sigma_{x \in X}\, \omega(x)\, [t(x) - s(x)]$$

$$= \langle \omega, t - s \rangle = \langle \omega, t \rangle - \langle \omega, s \rangle = 1 - 1 = 0 \quad ■$$

5. STRICT POSITIVITY

Henceforth, we assume that the following conditions hold.

5.1. Standing Assumption. $T \subseteq (\mathbb{R}^+)^X$ and $\Omega \subseteq V^+ \subseteq (\mathbb{R}^+)^X$ *are strictly positive on* $X \neq \emptyset$.

As a consequence of Assumption 5.1, we have $\Omega \neq \emptyset$, $e \neq 0$, and $\dim(V) = {}^{\#}X + 1 - \text{rank}(T)$ by Corollary 4.11.

5.2. Lemma. $0 \notin \text{aff}(T)$ and $\text{aff}(T) \cap (\mathbb{R}^+)^X$ is an antichain.

Proof. Let $S := \text{aff}(T) \cap (\mathbb{R}^+)^X$. By Theorem 4.14, $0 \notin S$, $V(T) = V(S)$, $e_T = e_S$, and $\Omega(T) = \Omega(S)$. Theorem 4.15 with T replaced by S implies that S is an antichain. ■

5.3. Lemma. There is a strictly positive element $\alpha \in \Omega$.

Proof. For each $x \in X$, select $\omega_x \in \Omega$ such that $0 < \omega_x(x)$. Let $n := {}^{\#}X$. Since Ω is convex, $\alpha := \Sigma_{x \in X}\, (1/n)\omega_x \in \Omega$ and it is obvious that α is strictly positive. ■

5.4. *Lemma.* If α is a strictly positive element of Ω and $\nu \in V$, there exists $\omega \in \Omega$ and there exist $a, b \in \mathbb{R}$ with $0 < a, b$ such that $\nu = a\alpha - b\omega$.

Proof. Assume the hypotheses and choose $a > 0$ such that $a > \max\{\nu(x)/\alpha(x) \mid x \in X\}$. Then $0 \neq a\alpha - \nu \in V \cap (\mathbb{R}^+)^X = V^+$. By Lemma 4.9, Ω is a cone base for V^+, so there exists $b > 0$ and there exists $\omega \in \Omega$ such that $b\omega = a\alpha - \nu$. ■

5.5. *Theorem.* (i) $V = V^+ - V^+$; (ii) $V = \text{lin}(\Omega)$; (iii) $\text{aff}(\Omega) = e^{-1}(1)$; (iv) $\dim(\Omega) = {}^{\#}X - \text{rank}(T)$.

Proof. (i) By Lemmas 5.3 and 5.4, $V = V^+ - V^+$. (ii) By Lemma 4.9, Ω is a cone base for V^+ and $V = V^+ - V^+$ part (i), whence $V = \text{lin}(\Omega)$. (iii) Because $\Omega \subseteq e^{-1}$ (1), we have $\text{aff}(\Omega) \subseteq e^{-1}$ (1). Conversely, suppose $\nu \in e^{-1}$ (1) and choose $a, b > 0$ and $\omega \in \Omega$ with $\nu = a\alpha - b\omega$ as in Lemma 5.4. Then $1 = e(\nu) = ae(\alpha) - be(\omega) = a - b$, whence $\nu = a\alpha + (1 - a)\omega \in \text{aff}(\Omega)$. (iv) Because e is a nonzero linear functional on V, (iii) implies that

$$\dim(\Omega) := \dim(\text{aff}(\Omega)) = \dim(e^{-1}(1)) = \dim(V) - 1$$

By Corollary 4.11, $\dim(V) = {}^{\#}X + 1 - \text{rank}(T)$, and it follows that $\dim(\Omega) = {}^{\#}X - \text{rank}(T)$. ■

By a *polyhedron*, we mean a subset of $\mathbb{R}^X$ that is an intersection of finitely many closed half-spaces; by a *polytope*, we mean a bounded polyhedron. A good reference for facts about polyhedra and polytopes is Grünbaum (1967).

5.6. *Theorem.* Ω is a nonempty polytope.

Proof. We have seen that V^+ is a polyhedral cone. Therefore, since $e^{-1}(1)$ is an affine subspace of V, it follows that $\Omega = V^+ \cap e^{-1}(1)$ is a polyhedron in $V \subseteq \mathbb{R}^X$. By Lemma 4.8, Ω is order bounded, whence norm bounded in $\mathbb{R}^X$, and a norm-bounded polyhedron is a polytope. ■

5.7. *Theorem.* The following conditions are mutually equivalent: (i) ${}^{\#}\Omega = 1$; (ii) $\text{rank}(T) = {}^{\#}X$; (iii) $\dim(V) = 1$; (iv) $\ker(e) = \{0\}$.

Proof. If ${}^{\#}\Omega = 1$, then $\dim(\Omega) = 0$, so $\text{rank}(T) = {}^{\#}X$ by part (iv) of Theorem 5.5, whence (i) $\Rightarrow$ (ii). That (ii) $\Rightarrow$ (iii) follows from the fact that $\dim(V) = {}^{\#}X + 1 - \text{rank}(T)$. Since e is a nonzero linear functional on V, it is clear that (iii) $\Rightarrow$ (iv). To prove (iv) $\Rightarrow$ (i), assume (iv) and suppose $\alpha, \omega \in \Omega$. Then $e(\alpha - \omega) = e(\alpha) - e(\omega) = 1 - 1 = 0$, so $\alpha - \omega \in \ker(e) = \{0\}$ and $\alpha = \omega$. ■

5.8. *Definition.* We denote the dual space of V by V^* and we define $V^{*+} \subseteq V^*$ to be the set of all positive linear functionals on V.

Evidently, V^{*+} is a cone in V^*, $e \in V^{*+}$, and all of the evaluation functionals f_x, $x \in X$, belong to V^{*+}. Since the evaluation functionals form a total subset of V^*, it follows that V^{*+} is a generating cone for V^*.

5.9. Theorem. e is an order-unit in V^*.

Proof. Let $f \in V^*$. By Theorem 5.6, Ω is compact; hence, $f(\Omega)$ is bounded in $\mathbb{R}$. Select $k \in \mathbb{Z}^+$ with $f(\omega) \leq k$ for all $\omega \in \Omega$ and let $0 \neq \nu \in V^+$. Since Ω is a cone base for V^+, there exist $a > 0$ and $\omega \in \Omega$ with $\nu = a\omega$, whence

$$f(\nu) = af(\omega) \leq ak = ake(\omega) = ke(a\omega) = ke(\nu)$$

and it follows that $f \leq ke$. ■

If $T \subseteq (\mathbb{Z}^+)^X$ is the total set of multiplicity functions for a finite effect algebra L, then V^* is a universal vector space for vector-valued measures on L (Foulis and Bennett, 1994). For $T \subseteq (\mathbb{R}^+)^X$, we have the following, more general result.

5.10. Theorem. Let W be a vector space over $\mathbb{R}$, let $w \in W$, and suppose that $\psi : X \to W$ satisfies $\Sigma_x\, t(x)\psi(x) = w$ for all $t \in T$. Then there is a unique linear transformation $\Psi : V^* \to W$ such that $\Psi(f_x) = \psi(x)$ for all $x \in X$. Furthermore, $\Psi(e) = w$.

Proof. Define $\phi : \mathbb{R}^X \to W$ by $\phi(y) := \Sigma_x\, y(x)\psi(x)$. By Theorem 4.10, $\mathbb{R}^X$ is the direct sum of V and $V^\perp = \text{lin}(T - t)$ for any choice of $t \in T$. Let $\eta : \mathbb{R}^X \to V$ be the projection of $\mathbb{R}^X$ onto V with $\ker(\eta) = V^\perp = \text{lin}(T - t)$. If $s \in T$, then

$$\phi(s - t) = \sum_x s(x)\psi(x) - \sum_x t(x)\psi(x) = w - w = 0$$

so $T - t \subseteq \ker(\phi)$, and it follows that $\ker(\eta) \subseteq \ker(\phi)$. Therefore, there exists a linear transformation $\Phi : V \to W$ such that $\phi = \Phi \circ \eta$. Let $g \in V^*$. Because V is a finite-dimensional inner-product space under the restriction of $\langle \cdot,\cdot \rangle$ to V, there is a unique $\gamma \in V$ such that $g(\nu) = \langle \gamma, \nu \rangle$ for all $\nu \in V$. Define $\Psi : V^* \to W$ by $\Psi(g) := \Phi(\gamma)$. For $x \in X$, let $\chi_x \in \mathbb{R}^X$ be the characteristic set function of $\{x\}$. Since the projection η is self-adjoint, we have

$$\langle \eta(\chi_x), \nu \rangle = \langle \chi_x, \eta(\nu) \rangle = \langle \chi_x, \nu \rangle = \nu(x) = f_x(\nu)$$

for all $\nu \in V$, so the linear functional $f_x \in V^*$ corresponds to the vector $\eta(\chi_x) \in V$, and it follows that

$$\Psi(f_x) = \Phi(\eta(\chi_x)) = \phi(\chi_x) = \psi(x)$$

The uniqueness of Ψ follows from the fact that $\{f_x | x \in X\}$ generates V^*. Also, for the linear functional $e \in V^*$, we have

$$e(\nu) = \langle t, \nu \rangle = \langle t, \eta(\nu) \rangle = \langle \eta(t), \nu \rangle$$

for all $\nu \in V$ and any choice of $t \in T$, whence

$$\Psi(e) = \Phi(\eta(t)) = \phi(t) = \sum_x t(x)\psi(x) = w \quad \blacksquare$$

6. EXTREME POINTS, FACES, AND FACETS OF Ω

We continue to assume that *both T and Ω are strictly positive on X.*

6.1. Lemma. If $Y \subseteq X$, then the evaluation functionals f_y, $y \in Y$, separate the points in Ω iff $\{f_y | y \in Y\} \cup \{e\}$ is a total set of linear functionals on V.

Proof. Suppose first that $\{f_y | y \in Y\} \cup \{e\}$ is total and let $\omega_1, \omega_2 \in \Omega$ with $f_y(\omega_1) = f_y(\omega_2)$ for all $y \in Y$. Then $\omega_1 - \omega_2 \in \ker(f_y)$ for all $y \in Y$ and $e(\omega_1 - \omega_2) = 1 - 1 = 0$, so $\omega_1 - \omega_2 \in \ker(e)$, and it follows that $\omega_1 - \omega_2 = 0$, so $\omega_1 = \omega_2$.

Conversely, suppose $\{f_y | y \in Y\}$ separates the points in Ω and let $\nu \in V$ with $f_y(\nu) = e(\nu) = 0$ for all $y \in Y$. By Lemma 5.4, we have $\nu = a\alpha - b\omega$ with $a, b > 0$ and $\alpha, \omega \in \Omega$. Thus $0 = e(\nu) = ae(\alpha) - be(\omega) = a - b$, so $\nu = a(\alpha - \omega)$, and it follows that $0 = f_y(\nu) = a(f_y(\alpha) - f_y(\omega))$, whence $f_y(\alpha) = f_y(\omega)$ for all $y \in Y$. Because $\{f_y | y \in Y\}$ separates the points of Ω, we have $\alpha = \omega$, so $\nu = a(\alpha - \omega) = 0$. $\blacksquare$

We denote the set of extreme points of a convex set Γ by $\text{ext}(\Gamma)$. By the theorem of Minkowski–Carathéodory, if Γ is a polytope, then $\text{ext}(\Gamma)$ is a finite set and $\Gamma = \text{con}(\text{ext}(\Gamma))$. The following theorem may be viewed as asserting that the extreme points of the polytope Ω are the probability weights with "small" supports.

6.2. Theorem (Rüttimann–Kläy). If $\omega \in \Omega$, then $\omega \in \text{ext}(\Omega)$ iff the set $\{f_y | y \in X \backslash \text{supp}(\omega)\}$ separates the points of Ω.

Proof. Let $Y := X \backslash \text{supp}(\omega)$. Suppose first that $\{f_y | y \in Y\}$ separates the points of Ω, $\omega = a\omega_1 + (1 - a)\omega_2$ with $0 < a < 1$ and $\omega_1, \omega_2 \in \Omega$. Then

$$y \in Y \Rightarrow 0 = \omega(y) = f_y(\omega) = af_y(\omega_1) + (1 - a)f_y(w_2)$$

and, owing to the fact that $0 \leq f_y(\omega_1), f_y(\omega_2)$, it follows that $y \in Y \Rightarrow f_y(\omega_1) = f_y(\omega_2) = 0$. But $\{f_y | y \in Y\}$ separates points of Ω, so $\omega_1 = \omega_2$, whence $\omega \in \text{ext}(\Omega)$.

Conversely, suppose $\omega \in \text{ext}(\Omega)$ and let $\nu \in V$ with $f_y(\nu) = e(\nu) = 0$ for all $y \in Y$. By Lemma 6.1, it will be sufficient to prove that $\nu = 0$. Let

$$a := \max\left\{\frac{|f_x(\nu)|}{f_x(\omega)} \;\middle|\; x \in \operatorname{supp}(\omega) = X\backslash Y\right\}$$

If $a = 0$, then $f_x(\nu) = 0$ for all $x \in X$, so $\nu = 0$ and we are done. Thus, we may assume that $a > 0$. For $x \in \operatorname{supp}(\omega)$, we have

$$|f_x(\nu)| \leq af_x(\omega) \tag{1}$$

and, owing to the fact that $f_y(\nu) = 0$ for all $y \in Y$, the inequality (1) holds for all $x \in X$. From (1) and the fact that $a > 0$, we have

$$0 \leq f_x(\omega) \pm (1/a)f_x(\nu) = f_x(\omega \pm (1/a)\nu)$$

for all $x \in X$, from which it follows that $\omega \pm (1/a)\nu \in V^+$. Since $e(\nu) = 0$, we also have $e(\omega \pm (1/a)\nu) = 1$, whence $\omega \pm (1/a)\nu \in \Omega$. But

$$\omega = \tfrac{1}{2}(\omega + (1/a)\nu) + \tfrac{1}{2}(\omega - (1/a)\nu)$$

contradicting $\omega \in \operatorname{ext}(\Omega)$. ∎

6.3. Definition. If Γ is a convex set and $\Lambda \subseteq \Gamma$, then Λ is a *face* of Γ iff for all $\gamma_1, \gamma_2 \in \Gamma$ and all $a \in \mathbb{R}$ with $0 < a < 1$,

$$a\gamma_1 + (1 - a)\gamma_2 \in \Lambda \Leftrightarrow \gamma_1, \gamma_2 \in \Lambda$$

Denote the set of all faces of Γ by $\mathcal{F}(\Gamma)$. A *proper* face of Γ is a face $\Lambda \in \mathcal{F}(\Gamma)$ with $\Lambda \neq \Gamma$. A *facet* of Γ is a maximal proper face of Γ. The *algebraic interior* of Γ is the set of all elements of Γ that belong to no proper face of Γ.

Thus a face of the convex set Γ is an extremal convex subset of Γ. Note that $\emptyset, \Gamma \in \mathcal{F}(\Gamma)$ and that the intersection of faces of Γ is again a face of Γ, so $\mathcal{F}(\Gamma)$ forms a complete lattice under set inclusion. The atoms in the lattice $\mathcal{F}(\Gamma)$ are precisely the singleton subsets $\{\gamma\}$ for $\gamma \in \operatorname{ext}(\Gamma)$ and the facets of Γ are the coatoms in $\mathcal{F}(\Gamma)$. If f is a linear functional, $r \in \mathbb{R}$, and $\gamma \in \Gamma \Rightarrow f(\gamma) \leq r$, then $f^{-1}(r) \cap \Gamma \in \mathcal{F}(\Gamma)$. A face having the latter form is said to be *exposed* by the linear functional f. In particular, for the polytope Ω, we have the following result.

6.4. Lemma. If $x \in X$, then $(f_x)^{-1}(1) \cap \Omega$ and $\ker(f_x) \cap \Omega$ are exposed faces of Ω.

Proof. For $\omega \in \Omega$, we have $f_x(\omega) \leq 1$ and $(-f_x)(\omega) \leq 0$, so the face $(f_x)^{-1}(1) \cap \Omega$ is exposed by f_x and the face $\ker(f_x) \cap \Omega$ is exposed by $-f_x$. ∎

If Γ is a polytope, then each face $\Lambda \in \mathcal{F}(\Gamma)$ is a polytope in its own right, and Λ is a facet of Γ iff $\dim(\Lambda) = \dim(\Gamma) - 1$. Every face of a polytope is the intersection of the facets that contain it, the algebraic interior of a polytope coincides with its topological interior as calculated in its affine span, and every nonempty polytope has a nonempty algebraic interior.

6.5. Theorem. The algebraic interior of Ω is the set of all strictly positive elements of Ω.

Proof. First suppose that $\alpha \in \Omega$ is strictly positive, but that α is not in the algebraic interior of Ω. Then there exists a proper face Λ of Ω with $\alpha \in \Lambda$. Let $\omega \in \Omega$ be an arbitrary probability weight. By Lemma 5.4, there are real numbers $a, b > 0$ and a probability weight $\omega' \in \Omega$ such that $\omega = a\alpha - b\omega'$. Thus $1 = e(\omega) = ae(\alpha) - be(\omega') = a - b$, so $a = b + 1 > 1$, $(1/a)\omega + (b/a)\omega' = \alpha \in \Lambda$ with $0 < 1/a < 1$, and $(1/a) + (b/a) = 1$. Because $\Lambda \in \mathcal{F}(\Omega)$, it follows that $\omega \in \Lambda$ and, since ω was arbitrary, $\Omega = \Lambda$, contradicting that fact that Λ is proper.

Conversely, suppose $\omega' \in \Omega$ and ω' is not strictly positive. Then there exists $x \in X$ such that $f_x(\omega') = 0$. By Lemma 6.4, $\Lambda := \ker(f_x) \cap \Omega \in \mathcal{F}(\Omega)$ with $\omega' \in \Lambda$. By Lemma 6.1, there exists a strictly positive $\alpha \in \Omega$. Thus $\alpha \notin \Lambda$, so Λ is a proper face of Ω, and it follows that ω' does not belong to the algebraic interior of Ω. ■

The following important theorem shows that every facet of Ω is exposed by the negative of an evaluation functional.

6.6. Theorem (Rüttimann–Kläy). If Δ is a facet of Ω, then there exists $x \in X$ such that $\Delta = \Omega \cap \ker(f_x)$.

Proof. Suppose $\Delta = \emptyset$. Since Δ is a facet, it follows that $\Omega = \{\alpha\}$ and, since there is a strictly positive probability weight in Ω, α must be strictly positive. Therefore, for any $x \in X$, $\alpha \notin \ker(f_x)$, so $\Omega \cap \ker(f_x) = \emptyset = \Delta$.

Thus, we may assume that $\Delta \neq \emptyset$, hence, that Δ has a nonempty algebraic interior. Select ω in the algebraic interior of Δ. Since $\omega \in \Delta$ and Δ is a proper face of Ω, it follows that ω is not in the algebraic interior of Ω, whence there exists $x \in X$ such that $f_x(\omega) = 0$ by Theorem 6.5. Let $\Lambda := \Omega \cap \ker(f_x)$, noting that Λ is a face of Ω and $\omega \in \Lambda$. Thus $\Delta \cap \Lambda$ is a face of Δ and $\omega \in \Delta \cap \Lambda$. Owing to the fact that ω is in the algebraic interior of Δ, it cannot belong to any proper face of Δ, and it follows that $\Delta \cap \Lambda = \Delta$, that is, $\Delta \subseteq \Lambda$. Since Δ is a maximal proper face of Ω, it follows that either $\Delta = \Lambda$ or $\Lambda = \Omega$. There is a strictly positive $\alpha \in \Omega$, and $\alpha \notin \ker(f_x)$, so $\alpha \notin \Lambda$, and therefore $\Delta = \Lambda = \Omega \cap \ker(f_x)$. ■

7. FACES AND SUPPORTS

Suppose that the elements of L represent propositions or effects associated with a physical system. Mielnik (1968), Ludwig (1983/85), and others have argued that the faces of the convex set of probability measures on L correspond to *properties or attributes* of the system. On the other hand, Foulis, *et al.* (1983) argued that such properties are represented by so-called *support sets*. Our purpose in the present section is to generalize the notion of a support set to our present context and to show how support sets are related to faces of Ω. In what follows, we continue to assume that *T and Ω are strictly positive on X.*

7.1. Definition. If Y is a nonempty subset of X and $f \in \mathbb{R}^X$, define $f_Y \in \mathbb{R}^Y$ to be the function obtained by restriction of f to Y. Also define $T_Y := \{t_Y | t \in T\}$.

Note that the mapping $f \mapsto f_Y$ is a linear surjection from $\mathbb{R}^X$ onto $\mathbb{R}^Y$ and that $f, g \in \mathbb{R}^X$ with $f \leq g \Rightarrow f_Y \leq g_Y$ in $\mathbb{R}^Y$.

7.2. Definition. If Y is a nonempty subset of X, then Y is a *T-support* iff T_Y is an antichain. By special convention, the empty set $\emptyset \subseteq Y$ is also regarded as a *T-support*. The set of all T-supports is denoted by $\mathcal{S}(T)$.

If T is understood, we denote $\mathcal{S}(T)$ by $\mathcal{S}$. Obviously, the union of T-supports is again a T-support, whence $\mathcal{S}$ is a lattice under inclusion.

7.3. Lemma. $\omega \in \Omega \Rightarrow \text{supp}(\omega) \in \mathcal{S}$.

Proof. Let $Y := \text{supp}(\omega)$. We have to prove that T_Y is an antichain in $\mathbb{R}^Y$. Suppose $s, t \in T$, $s_Y \leq t_Y$. If $y \in Y = \text{supp}(\omega)$, then $0 < \omega(y)$ and $t_Y(y) - s_Y(y) \geq 0$, whence $0 \leq \omega(y)[t_Y(y) - s_Y(y)]$. Also, $\omega(x) = 0$ for $x \in X \backslash Y$, and it follows that

$$\Sigma_{y \in Y}\, \omega(y)[t_Y(y) - s_Y(y)] = \Sigma_{x \in X}\, \omega(x)[t(x) - s(x)]$$

$$= \langle \omega, t \rangle - \langle \omega, s \rangle = e(t) - e(s) = 1 - 1 = 0$$

hence each summand satisfies $\omega(y)[t_Y(y) - s_Y(y)] = 0$. Since $0 < \omega(y)$, it follows that $t_Y(y) - s_Y(y) = 0$ for all $y \in Y$, whence $s_Y = t_Y$ and T_Y is an antichain. ■

7.4. Definition. If $\Lambda \subseteq \Omega$ and $Y \subseteq X$, then:

(i) $\text{supp}(\Lambda) := \cup_{\omega \in \Lambda} \text{supp}(\omega)$.
(ii) $\text{face}\,(Y) := \{\omega \in \Omega | \text{supp}(\omega) \subseteq Y\}$.

7.5. Theorem. If $\Lambda, \Lambda_1, \Lambda_2, \ldots, \Lambda_k \subseteq \Omega$ and $Y, Y_1, Y_2, \ldots, Y_k \subseteq X$, then:

(i) $\text{supp}\,(\Lambda) \in \mathcal{S}$ and $\text{face}(Y) = \cap \{\Omega \cap \ker(f_x) | x \in X \backslash Y\} \in \mathcal{F}(\Omega)$.
(ii) $\Lambda_1 \subseteq \Lambda_2 \Rightarrow \text{supp}(\Lambda_1) \subseteq \text{supp}(\Lambda_2)$ and $Y_1 \subseteq Y_2 \Rightarrow \text{face}(Y_1) \subseteq \text{face}(Y_2)$.
(iii) $\text{supp}(\Lambda) \subseteq Y \Leftrightarrow \Lambda \subseteq \text{face}(Y)$.

(iv) $\Lambda \subseteq \text{face}(\text{supp}(\Lambda))$ and $\text{supp}(\text{face}(Y)) \subseteq Y$.
(v) $\text{supp}(\Lambda) = \text{supp}(\text{face}(\text{supp}(\Lambda)))$ and $\text{face}(Y) = \text{face}(\text{supp}(\text{face}(Y)))$.
(vi) $\text{supp}(\cup_i \Lambda_i) = \cup_i \text{supp}(\Lambda_i)$ and $\text{face}(\cap_i Y_i) = \cap_i \text{face}(Y_i)$.

Proof. (i) That $\text{supp}(\Lambda) \in \mathcal{S}$ follows from Lemma 7.3 and the fact that $\mathcal{S}$ is closed under unions. That $\text{face}(Y) = \cap\{\Omega \cap \ker(f_x) | x \in X \setminus Y\}$ is clear. By Lemma 6.4, each $\Omega \cap \ker(f_x)$ is an exposed face of Ω and the intersection of faces is a face, so $\text{face}(Y) \in \mathcal{F}(\Omega)$. Part (ii) is obvious and part (iii), which is also obvious, shows that $\Lambda \mapsto \text{supp}(\Lambda)$ is a residuated mapping and $Y \mapsto \text{face}(Y)$ is its dual residual mapping (Blyth and Janowitz, 1972). Parts (iv)–(vi) can be checked directly, although they follow immediately from the theory of residuated mappings. ■

7.6. Lemma. If Δ is a facet of Ω, there exists $x \in X$ such that $\Delta = \text{face}(X \setminus \{x\})$.

Proof. By Theorem 6.6, there exists $x \in X$ such that

$$\begin{aligned}\Delta = \ker(f_x) \cap \Omega &= \{\omega \in \Omega | x \notin \text{supp}(\omega)\} \\ &= \{\omega \in \Omega | \text{supp}(\omega) \subseteq X \setminus \{x\}\} = \text{face}(X \setminus \{x\}) \quad ■\end{aligned}$$

7.7. Theorem. If $\Lambda \in \mathcal{F}(\Omega)$, then $\Lambda = \text{face}(\text{supp}(\Lambda))$.

Proof. Evidently, $\emptyset = \text{face}(\emptyset)$ and $\emptyset = \text{supp}(\emptyset)$; therefore $\emptyset = \text{face}(\text{supp}(\emptyset))$, so we may assume that $\Lambda \neq \emptyset$. Thus there are facets $\Delta_1, \Delta_2, \ldots, \Delta_k \in \mathcal{F}(\Omega)$ such that $\Lambda = \Delta_1 \cap \Delta_2 \cap \ldots \cap \Delta_k$. By Lemma 7.6, there exist $Y_1, Y_2, \ldots, Y_k \subseteq X$ such that $\Delta_i = \text{face}(Y_i)$ for $i = 1, 2, \ldots, k$. Let $Y := Y_1 \cap Y_2 \cap \ldots \cap Y_k$. Then $\Lambda = \text{face}(Y)$ by part (vi) of Theorem 7.5, so by part (v) of Theorem 7.5, $\Lambda = \text{face}(\text{supp}(\text{face}(Y))) = \text{face}(\text{supp}(\Lambda))$. ■

7.8. Corollary. If $\Lambda_1, \Lambda_2 \in \mathcal{F}(\Omega)$ then $\text{supp}(\Lambda_1) \subseteq \text{supp}(\Lambda_2) \Rightarrow \Lambda_1 \subseteq \Lambda_2$.

7.9. Corollary. If $\Lambda \subseteq \Omega$, then $\text{face}(\text{supp}(\Lambda))$ is the smallest face of Ω that contains Λ.

Proof. By parts (i) and (iv) of Theorem 7.5, $\Lambda \subseteq \text{face}(\text{supp}(\Lambda)) \in \mathcal{F}(\Omega)$. Suppose $\Lambda \subseteq \Delta \in \mathcal{F}(\Omega)$. Then by part (ii) of Theorem 7.5 and Theorem 7.7, $\text{face}(\text{supp}(\Lambda)) \subseteq \text{face}(\text{supp}(\Delta)) = \Delta$. ■

7.10. Corollary. If $\omega \in \Omega$, then the following conditions are mutually equivalent: (i) $\omega \in \text{ext}(\Omega)$; (ii) $\{\omega\} \in \mathcal{F}(\Omega)$; (iii) $\{\omega\} = \text{face}(\text{supp}(\omega))$; (iv) $\omega' \in \Omega$, $\text{supp}(\omega') \subseteq \text{supp}(\omega) \Rightarrow \omega' = \omega$.

Proof. That (i) $\leftrightarrow$ (ii) is clear and (ii) $\Leftrightarrow$ (iii) by Corollary 7.9. Since $\text{face}(\text{supp}(\omega)) = \{\omega' \in \Omega | \text{supp}(\omega') \subseteq \text{supp}(\omega)\}$, we have (iii) $\Leftrightarrow$ (iv). ■

7.11. Corollary. Every nonempty face of Ω has the form face(supp(ω)) for some $\omega \in \Omega$.

Proof. Suppose $\emptyset \neq \Gamma \in \mathcal{F}(\Omega)$. Then Γ is a nonempty face of the polytope Ω, whence Γ itself is a nonempty polytope, so there exists ω in the algebraic interior of Γ. Because ω belongs to no proper face of Γ, it follows that Γ is the smallest face of Ω that contains ω, whence Γ = face(supp(ω)) by Corollary 7.9. ■

If $\Lambda \in \mathcal{F}(\Omega)$, then supp($\Lambda$) $\in \mathcal{S}$ by part (i) of Theorem 7.5, so Theorem 7.7 shows that $Y \mapsto$ face(Y) maps the support lattice $\mathcal{S}$ onto the face lattice $\mathcal{F}(\Omega)$. On the other hand, although the mapping $\Lambda \mapsto$ supp(Λ) from $\mathcal{F}(\Omega)$ to $\mathcal{S}$ is injective by Corollary 7.8, simple examples show that it need not map the face lattice $\mathcal{F}(\Omega)$ onto the support lattice $\mathcal{S}$. In fact, it maps $\mathcal{F}(\Omega)$ onto the subset of $\mathcal{S}$ consisting of the so-called *stochastic supports*.

7.12. Definition. A support of the form supp(Λ) for $\Lambda \subseteq \Omega$ is called a *stochastic support*.

7.13. Lemma. If $\emptyset \neq Y \in \mathcal{S}$, then the following conditions are mutually equivalent: (i) Y is a stochastic support; (ii) Y = supp(face(Y)); (iii) there exists $\emptyset \neq \Gamma \in \mathcal{F}(\Omega)$ with Y = supp(Γ); (iv) Y = supp(ω) for some $\omega \in$ face(Y).

Proof. (i) $\Rightarrow$ (ii): If $\Lambda \subseteq \Omega$ and Y = supp(Λ), then part (v) of Theorem 7.5 implies that Y = supp(face(supp(Λ))) = supp(face(Y)). That (ii) $\Rightarrow$ (iii) follows from the fact that face(Y) $\in \mathcal{F}(\Omega)$.

(iii) $\Rightarrow$ (iv): If $\emptyset \neq \Gamma \in \mathcal{F}(\Omega)$ with Y = supp(Γ), Corollary 7.11 implies that Γ = face(supp(ω)) for some $\omega \in \Omega$, whence Y = supp(Γ) = supp(face(supp(ω))) = supp(ω). Also, $\omega \in$ face(supp(ω)) = face(Y). That (iv) $\Rightarrow$ (i) is obvious. ■

In the following theorem and its proof, we use the notation of Definition 7.1.

7.14. Theorem (Rüttimann–Kläy). Let $\emptyset \neq \Gamma \in \mathcal{F}(\Omega)$ and let $Y :=$ supp(Γ) be the corresponding stochastic support. Then there is an order-preserving vector space isomorphism Φ: $V(T_Y) \to$ lin(Γ) $\subseteq$ V such that $\Phi^{-1}(\nu) = \nu_Y$ for $\nu \in$ lin(Γ). Furthermore, $\Phi(\Omega(T_Y)) = \Gamma$ and $e \circ \Phi$ is the intensity functional for $V(T_Y)$.

Proof. Define Φ: $V(T_Y) \to \mathbb{R}^X$ by

$$\Phi(\eta)(x) := \begin{cases} \eta(x) & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases} \qquad \text{for} \quad \eta \in V(T_Y) \text{ and } x \in X$$

Clearly, Φ is linear, injective, order preserving, and $\Phi(\eta) = \nu \Rightarrow \eta = \nu_Y$.

Let $\eta \in V(T_Y)$, $\nu = \Phi(\eta)$, and $t \in T$. Then, owing to the fact that $\nu(x) = 0$ for $x \notin Y$, we have

$$\langle \eta, t_Y \rangle = \sum_{y \in Y} \eta(y)t(y) = \sum_{x \in X} \nu(x)t(x) = \langle \nu, t \rangle \tag{1}$$

Because $\langle \eta, t_Y \rangle$ is independent of the choice of $t \in T$, (1) implies that $\nu = \Phi(\eta) \in V$, so

$$\Phi(V(T_Y)) \subseteq V \tag{2}$$

Furthermore, by (1), $(e \circ \Phi)(\eta) = e(\nu) = \langle \nu, t \rangle = \langle \eta, t_Y \rangle$, so $e \circ \Phi$ is the intensity functional for $V(T_Y)$.

If $\eta \in \Omega(T_Y) = V(T_Y) \cap (\mathbb{R}^+)^Y \cap (e \circ \Phi)^{-1}(1)$, then, by (2), $\Phi(\eta) \in V \cap (\mathbb{R}^+)^X = V^+$ and $e(\Phi(\eta)) = 1$, whence $\Phi(\eta) \in \Omega$. Thus,

$$\Phi(\Omega(T_Y)) \subseteq \Omega \tag{3}$$

Clearly, $\text{supp}(\Phi(\Omega(T_Y))) \subseteq Y$, so, by (3),

$$\Phi(\Omega(T_Y)) \subseteq \text{face}(\text{supp}(\Phi(\Omega(T_Y))) \subseteq \text{face}(Y) = \Gamma \tag{4}$$

By Lemma 7.13, there exists $\omega \in \text{face}(Y) = \Gamma$ with $Y = \text{supp}(\omega)$. Evidently, $\omega_Y \in \Omega(T_Y)$ and ω_Y is strictly positive on Y, so $\Omega(T_Y)$ is a strictly positive set of T_Y-probability weights. Therefore, $V(T_Y) = \text{lin}(\Omega(T_Y))$ by part (ii) of Theorem 5.5. Consequently, by (4),

$$\Phi(V(T_Y)) = \text{lin}(\Phi(\Omega(T_Y))) \subseteq \text{lin}(\Gamma) \tag{5}$$

Suppose $\gamma \in \Gamma$. Then $\text{supp}(\gamma) \subseteq \text{supp}(\Gamma) = Y$ and $\gamma \in \Gamma \subseteq \Omega \subseteq (\mathbb{R}^+)^X$, so $\gamma_Y \in (\mathbb{R}^+)^Y$. If $t \in T$, then the fact that $\gamma(x) = 0$ for $x \in X \setminus Y$ implies that

$$1 = \langle \gamma, t \rangle = \sum_{x \in X} \gamma(x)t(x) = \sum_{y \in Y} \gamma(y)t(y) = \langle \gamma_Y, t_Y \rangle$$

whence $\gamma_Y \in \Omega(T_Y) \subseteq V(T_Y)$. Evidently, $\Phi(\gamma_Y) = \gamma$, and it follows that $\Gamma \subseteq \Phi(\Omega(T_Y))$, so in view of (4), we have

$$\Phi(\Omega(T_Y)) = \Gamma \tag{6}$$

By (6),

$$\text{lin}(\Gamma) = \text{lin}(\Phi(\Omega(T_Y)) = \Phi(\text{lin}(\Omega(T_Y)) = \Phi(V(T_Y))$$

so Φ is a linear isomorphism from $V(T_Y)$ onto $\text{lin}(\Gamma)$. ■

8. CALCULATING THE EXTREME POINTS OF Ω

We maintain our standing assumption that *T and Ω are strictly positive on X* and turn our attention to the problem of calculating the extreme points in Ω.

8.1. Theorem. Let $\omega \in \Omega$ with $Y = \text{supp}(\omega)$. Then the following conditions are mutually equivalent:

(i) $\omega \in \text{ext}(\Omega)$.
(ii) There is a unique T_Y-probability weight (namely ω_Y).
(iii) $\text{rank}(T_Y) = {}^{\#}Y$.

Proof. Let $\Gamma := \text{face}(Y) = \text{face}(\text{supp}(\omega))$ in Theorem 7.14. By Corollary 7.10, $\omega \in \text{ext}(\Omega)$ iff $\Gamma = \{\omega\}$ and by Theorem 7.14, $\omega \in \Gamma = \Phi\ (\Omega(T_Y))$, whence (i) $\Leftrightarrow$ (ii). That (ii) $\Leftrightarrow$ (iii) follows from Theorem 5.7 with X replaced by Y and T replaced by T_Y. ■

For simplicity in what follows, let $X = \{1, 2, \ldots, n\}$ and identify each function $f \in \mathbb{R}^X$ with the corresponding vector

$$f = (f_1, f_2, \ldots, f_n) := (f(1), f(2), \ldots, f(n))$$

in the coordinate Euclidean space $\mathbb{R}^n$. Thus $T, V, \Omega \subseteq \mathbb{R}^n$. By Theorem 4.14, we may replace T by a maximal linearly independent subset of T without affecting V, Ω, or e, and *we assume that this has been done*. Thus, in what follows,

$$m := {}^{\#}T = \text{rank}(T) \leq n \qquad \text{and} \qquad \dim(V) = n + 1 - m \tag{1}$$

For purposes of calculation, it is convenient to represent $T \subseteq \mathbb{R}^n$ as an $m \times n$ matrix with the vectors in T as its rows. (The order in which these rows are arranged is of no significance.) We also denote this matrix by $T = [t_{ij}]$, since one can tell from the context whether the set T or the matrix T is intended. Note that the rank of the set T coincides with the rank of the matrix T.

The column vectors of the matrix T are understood to be labeled by the elements of $X = \{1, 2, \ldots, n\}$. Thus, if $Y \subseteq X$, then the set T_Y corresponds to a truncated matrix, also denoted by T_Y, obtained by removing from the matrix T all column vectors labeled by $j \in X \setminus Y$.

8.2. Theorem. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in (\mathbb{R}^+)^n$ with $Y = \text{supp}(\omega) = \{j \in X | \omega_j \neq 0\}$. Then $\omega \in \text{ext}(\Omega)$ iff there exists $J \subseteq X$ with ${}^{\#}J = m$ such that the $m \times m$ matrix T_J is nonsingular, $Y \subseteq J$, and $\Sigma_{j \in J}\, t_{ij}\omega_j = 1$ for $i = 1, 2, \ldots, m$.

Proof. Suppose $\omega \in \text{ext}(\Omega)$. Then part (iii) of Theorem 8.1 implies that the column vectors in T labeled by $j \in Y$ are linearly independent, whence these column vectors can be extended to a maximal set of linearly independent column vectors labeled, say, by $j \in J \subseteq X$. Thus, ${}^{\#}J = \text{rank}(T) = m$ and the $m \times m$ matrix T_J is nonsingular. Since $\omega_j = 0$ for $j \notin J$ and $\Sigma_{j \in X}\, t_{ij}\omega_j = 1$ for $i = 1, 2, \ldots, m$, it follows that $\Sigma_{j \in J}\, t_{ij}\omega_j = 1$ for $i = 1, 2, \ldots, m$. The converse also follows in an obvious way from Theorem 8.1. ■

8.3. Theorem (Greechie–Miller). If the vectors in T (or the row vectors in the matrix T) have only rational entries, then each vector $\omega \in \text{ext}(\Omega)$ has only rational entries.

Proof. If the nonsingular matrix T_J in Theorem 8.2 has only rational entries, then the system of m linear equations $\Sigma_{j \in J}\, t_{ij}\omega_j = 1$ will have a unique solution ω_j, $j \in J$, consisting only of rational numbers. Since $\omega_j = 0$ for $j \notin J$, it follows that ω has only rational entries. ■

As a corollary of Theorem 8.3, if a finite effect algebra L carries enough probability measures μ so that each nonzero element p of L has strictly positive probability $0 < \mu(p)$ for some μ, then the convex set of all probability measures on L forms a rational polytope, i.e., a polytope with extreme points having only rational coordinates.

Theorem 8.2 provides a basis for the following algorithm for computing $\text{ext}(\Omega)$ given a strictly positive antichain $T \subseteq (\mathbb{R}^+)^n$.

8.4. Algorithm. Form an $m \times n$ matrix $[t_{ij}]$ of rank m, the rows of which constitute a maximal linearly independent subset of the vectors in the original set T. For each m-element subset J of $\{1, 2, \ldots, n\}$, form the square $m \times m$ submatrix T_J of $[t_{ij}]$ consisting of the columns labeled by $j \in J$. Check each T_J for nonsingularity. If T_J is nonsingular, find the unique solution ω_j, $j \in J$, of the system of linear equations $\Sigma_{j \in J}\, t_{ij}\omega_j = 1$ for $i = 1, 2, \ldots, m$. If $0 \leq \omega_j$ for all $j \in J$, then the vector $\omega_J := (\omega_1, \omega_2, \ldots, \omega_n)$ with $\omega_k = 0$ for $k \notin J$ is an extreme point of ω. Every extreme point of Ω will show up as an ω_J.

Using standard methods of matrix algebra, Algorithm 8.4 is easily implemented on a computer, and it yields a reasonable technique for calculating $\text{ext}(\Omega)$ provided that n is not too large. However, the calculation requires examining $n!/[(n - m)!\, m!]$ square $m \times m$ submatrices T_J corresponding to the m-element subsets J of $X = \{1, 2, \ldots, n\}$, so it rapidly becomes impractical for larger values of n. We commend to others with more sophisticated programming talents than ours the problem of improving Algorithm 8.4.

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